Asynchronous DeGroot Dynamics

ron.peretz@biu.ac.il

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DGR
Ran P.
Dor Elboim
Galit
Ashkenzi
-Golan

Yuval Peres

DeGroot Dynamics

- $G = (V, E)$ — undirected locally finite graph
- Agents are vertices
- $A_t \in \mathbb{R}^V$ —opinions at time t
- A_0 —initial opinions
- updating rule:
- $P : \mathbb{R}^V \to \mathbb{R}^V$
- $$(PA)(v) = \frac{1}{\deg(v)} \sum_{w \in N_v} A(v)$$

Timing

- Synchronous:

$$t = 0,1,2,\ldots,\ \ A_{t+1} = PA_t$$

- Asynchronous:

 iid Poisson clocks on vertices

$$A_t(v) = \begin{cases} (PA_{t-})(v) & v \text{ rings at time } t \\ A_{t-}(v) & \text{otherwise} \end{cases}$$

Known facts (synchronous timing)

- Finite graph: $\exists \lim_{t\to\infty} A_{2t}$. Furthermore,

  G non-bipartite $\Rightarrow \lim_{t\to\infty} A_t = \sum_v \pi_v A_0(v)$

- Infinite graph, bounded degree, bounded iid initial opinions, expectation μ

$$\forall v \quad \lim_{t\to\infty} A_t(v) = \mu,$$ almost surely.

DGR results (asynchronous timing)

- Finite graph. There exists a r.v. C s.t.

$$\forall v \quad \lim_{t\to\infty} A_t(v) = C, \text{ almost surely.}$$

- Further, iid initial opinions variance σ^2, then

$$\mathrm{Var}(C) = \mathcal{O}(\pi_{\max}\sigma^2)\,.$$

- Infinite graph. Bounded degree, iid initial opinions, finite variance, then

$$\forall v \quad \lim_{t\to\infty} A_t(v) = \mu, \text{ in probability.}$$

DRY results (asynchronous timing)

- Infinite graph. Bounded degree, bounded iid initial opinions.

$$\forall v \quad \lim_{t\to\infty} A_t(v) = \mu, \text{ almost surely.}$$

- Finite graph. Consensus convergence rate.

$$\tau_\epsilon := \min\{t : \forall v, w \; A_t(v) - A_t(w) \le \epsilon\},$$

$$\mathbb{E}[\tau_\epsilon] \le \begin{cases} 4 \cdot \mathrm{diam}(G) \cdot |E| \cdot \lceil \log_2(1/\epsilon) \rceil, \\ \log(2|E|/\epsilon^2)/\mathrm{spectral-gap}(G), \\ \mathcal{O}_{\epsilon,\deg(G)}(\log^{20}(|V|)). \end{cases}$$

Convergence to consensus in finite graphs

- Dirichlet energy:

$$\mathscr{E}(A) := \frac{1}{2|E|} \sum_{vw \in E} (A(v) - A(w))^2 = \langle (I - P)A, A \rangle_\pi$$

- $\mathscr{E}(A_t) \searrow 0, \qquad \max_v (A_t(v)) \searrow C.$

- More elaborate arguments provide the rate of convergence.

Consensus variance — finite graphs

- Consider $\mu_t := \langle A_t, 1 \rangle_\pi$, martingale, $\mu_t \to C$, $\mathrm{Var}(\mu_t) \to \mathrm{Var}(C)$.

$$\begin{aligned}\mathbb{E}[(\mu_{t+h} - \mu_t)^2 \mid \mathcal{F}_t] &= h \sum_v \pi_v^2 (PA_t(v) - A_t(v))^2 + \mathcal{O}(h^2) \\ &\leq \pi_{\max} \cdot h \,||(I - P)A_t||^2_{L_2(\pi)} + \mathcal{O}(h^2) \\ &= \pi_{\max} \cdot \mathbb{E}[\mathcal{E}(A_t) - \mathcal{E}(A_{t+h}) \mid \mathcal{F}_t] + \mathcal{O}(h^2).\end{aligned}$$

$$\mathrm{Var}(\mu_t) = \mathrm{Var}(\mu_0) + \pi_{\max} \left(E[\mathcal{E}(A_0)] - E[\mathcal{E}(A_t)] \right) \to \left(\sum_v \pi_v^2 + \pi_{\max} \right) \sigma^2$$

Backward looking approach — fragmentation

- Fix $o \in V$. Let X_t be RW on G originating at o.
- $\mathcal{F}$ — the σ-algebra generated by clock rings.
- $m_t(v) := \mathbb{P}(X_t = v \,|\, \mathcal{F})$.
- Observation:

$$A_t(o) \sim \sum_v m_t(v) A_0(v),$$

m, A_0 independent.

Convergence in probability — infinite graphs

- $$\mathrm{Var}(A_t(o)) = \sum_v \mathbb{E}[(m_t(v))^2]\sigma^2$$
- **Proposition.**

$$\sum_v \mathbb{E}[(m_t(v))^2] = \mathcal{O}\left(\frac{\deg(G)}{\sqrt{t}}\right).$$

Proof of the proposition

- X_t^1, X_t^2 two RWs originating at o, same Poisson cloaks, independent trajectories.
- Observation. $\mathbb{E}[m_t(v)^2] = \mathbb{P}(X_t^1 = X_t^2 = v)$.
- $\sum_v \mathbb{E}[(m_t(v))^2] = \mathbb{P}(X_t^1 = X_t^2) = ?$

Proof of the proposition

- $\hat{X}^1_n, \hat{X}^2_n$ — resp. (independent) trajectories
- $N_1(t), N_2(t)$— resp. Poisson jumps
- I.e., $X^i(t) = \hat{X}^i_{N_i(t)}$.

Proof of the proposition

$$\mathbb{P}(X_t^1 = X_t^2) = \sum_n \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1, p_{n_2}^{-1} \prec \hat{X}^2, N_1(t) = n_1, N_2(t) = n_2)$$

$$= \sum_n \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1)\mathbb{P}(p_{n_2}^{-1} \prec \hat{X}^2)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p_{n_1} \prec \hat{X}^1, p_{n_2}^{-1} \prec \hat{X}^2)$$

$$= \sum_n \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1)\mathbb{P}(p_{n_2}^{-1} \prec \hat{X}^2)$$
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Proof of the proposition

$$\ldots = \sum_{n} \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1)\mathbb{P}(p_{n_2}^{-1} \prec \hat{X}^2)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$$\leq \sum_{n} \sum_{p \in C_n} \sum_{n_1+n_2=n} \frac{\deg(G)}{\deg(o)} \mathbb{P}(p \prec \hat{X}^1)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$$= \frac{\deg(G)}{\deg(o)} \sum_{n} \sum_{p \in C_n} \mathbb{P}(p \prec \hat{X}^1)$$
$$\mathbb{P}(N_1(t) + N_2(t) = n \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

Proof of the proposition

$$\ldots = \sum_{n} \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1)\mathbb{P}(p_{n_2}^{-1} \prec \hat{X}^2)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$$\leq \sum_{n} \sum_{p \in C_n} \sum_{n_1+n_2=n} \frac{\deg(G)}{\deg(o)} \mathbb{P}(p \prec \hat{X}^1)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$$= \frac{\deg(G)}{\deg(o)} \sum_{n} \sum_{p \in C_n} \mathbb{P}(p \prec \hat{X}^1) = O\left(\frac{\deg(o)}{\sqrt{n}}\right)$$
$$\mathbb{P}(N_1(t) + N_2(t) = n \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

Proof of the proposition

$$\ldots = \sum_n \sum_{p \in C_n} \sum_{n_1+n_2=n} \mathbb{P}(p_{n_1} \prec \hat{X}^1)\mathbb{P}(p_{n_2}^{-1} \prec \hat{X}^2)$$
$$\mathbb{P}(N_1(t) = n_1, N_2(t) = n_2 \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$$\leq \sum_n \sum_{p \in C_n} \sum_{n_1+n_2=n} \frac{\deg(G)}{\deg(o)} \mathbb{P}(p \prec \hat{X}^1)$$
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$$= \frac{\deg(G)}{\deg(o)} \sum_n \sum_{p \in C_n} \mathbb{P}(p \prec \hat{X}^1)$$
$$\mathbb{P}(N_1(t) + N_2(t) = n \,|\, p \prec \hat{X}^1, p^{-1} \prec \hat{X}^2)$$

$= O\left(\frac{\deg(o)}{\sqrt{n}}\right)$

$= O(P(N(2t)=n))$